

INFINITELY MANY SMALL EXOTIC STEIN FILLINGS

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ABSTRACT. We show that there exist infinitely many simply connected compact Stein 4-manifolds with $b_2 = 2$ such that they are all homeomorphic but mutually non-diffeomorphic, and they are Stein fillings of the same contact 3-manifold on their boundaries. We also describe their handlebody pictures.

1. INTRODUCTION

Stein 4-manifolds enjoy some rigidity properties which are useful studying their topology. For example, the Stein manifolds, which certain closed 3-manifolds bound, are unique up to diffeomorphisms (cf. [12], [14]). Hence identifying exotic Stein 4-manifolds is a particularly interesting problem. In [7] by using Lefschetz fibrations on knot surgered elliptic surfaces, Akhmedov-Etnyre-Mark-Smith constructed infinitely many exotic (i.e. all homeomorphic but mutually non-diffeomorphic) simply connected compact Stein fillings of Seifert fibered contact 3-manifolds. Recently Akhmedov-Ozbagci [8] generalized this example to a larger family of Seifert fibered contact 3-manifolds. These 4-manifolds have large second Betti numbers, so finding small concrete examples of such manifolds and describing their handlebody pictures is a natural question in 4-manifold topology. In [6], generalization of the cork twisting constructions of [4], the authors constructed arbitrary many exotic compact Stein 4-manifolds which have the same topological invariants of a given 2-handlebody X with $b_2(X) \geq 1$, though we do not know whether these exotic Stein manifolds induce the same contact structure on their boundaries.

Here we construct infinitely many exotic Stein fillings of a fixed contact 3-manifold with the small second betti number. We also give their Stein handlebody pictures.

Theorem 1.1. *There exist infinitely many simply connected compact Stein 4-manifolds with $b_2 = 2$ such that they are all homeomorphic but mutually non-diffeomorphic, furthermore they are Stein fillings of the same contact 3-manifold.*

The construction is inspired by a recent paper [1] of the first author, where multiple log transforms produced infinitely many Stein fillings which are mutually exotic rel boundary. Here instead of multiple log transforms, we use single log transforms for the construction. This result should be contrasted with the result of [16], which says that any p -log transform ($p > 1$) along a homologically non-torsion c -embedded torus never produces a compact 4-manifold which admits a Stein structure.

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2. CONSTRUCTION

Let X be the compact smooth 4-manifold given by the left handlebody picture in Figure 1. Note that X contains an obvious $T^2 \times D^2$. Then by performing X the p -log transform operation in its interior we get X_p , which is the right picture. Here we used the description of p -log transform operation given in Section 4 of [3].

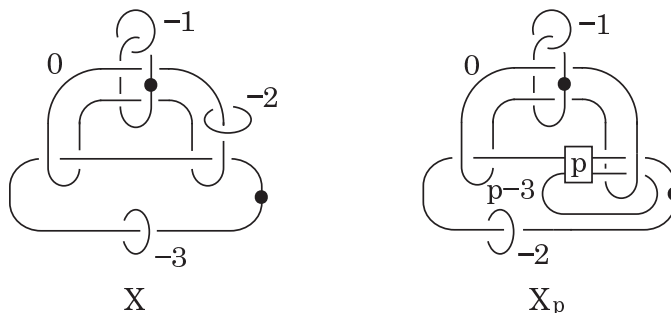


FIGURE 1. X and X_p

Note that X and X_p are simply connected and that their intersection forms are given by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & -2p^2 + p - 3 \end{pmatrix},$$

respectively. The intersection forms of X and X_p are thus unimodular and indefinite, which shows that ∂X is a homology 3-sphere. It is easy to see that the form of X_p is even if and only if p is odd. Therefore, Boyer’s theorem [9] together with the classification of the unimodular indefinite forms tells the following.

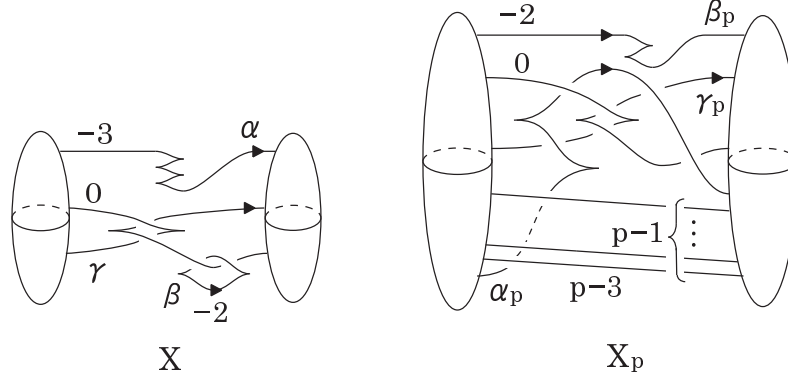
Lemma 2.1. (1) X_p and X_q are homeomorphic to each other if and only if the parities of p and q coincide.

(2) X_p is homeomorphic to X if and only if p is a positive odd integer.

Canceling the upper 1- and 2-handle pairs and converting the 1-handle notations, we get the Stein handlebody pictures of X and X_p in Figure 2 (For the time being, ignore symbols $\alpha, \beta, \gamma, \dots$). It follows from Eliashberg’s theorem [11] that X and X_p ($p \geq 1$) admit Stein structures. Rest of the paper, we equip X and X_p with Stein structures given by these Legendrian pictures.

3. DETECTING SMOOTH STRUCTURES

Next we detect smooth structures by applying the adjunction inequality. The argument is a simplification of the genus arguments of [5] and [16].

FIGURE 2. Stein handlebodies of X and X_p

Let α, β, γ and $\alpha_p, \beta_p, \gamma_p$ be the oriented attaching circles (framed links) of the 2-handles of X and X_p as indicated in Figure 2. The rotation numbers of these circles are as follows.

$$r(\alpha) = 2, \quad r(\beta) = 0, \quad r(\gamma) = 0, \quad r(\alpha_p) = -1, \quad r(\beta_p) = 1, \quad r(\gamma_p) = 0.$$

Define the basis S, T of $H_2(X; \mathbb{Z})$ and the basis R_p, T_p of $H_2(X_p; \mathbb{Z})$ given by 2-handles as follows.

$$S = [\beta], \quad T = [\gamma], \quad R_p = [\alpha_p - p\beta_p], \quad T_p = [\gamma_p].$$

Furthermore define the class $S_p \in H_2(X_p; \mathbb{Z})$ as follows.

$$S_p = \begin{cases} R_{2q-1} + ((2q-1)^2 - q + 1)T_{2q-1}, & \text{if } p = 2q-1 \text{ for some integer } q; \\ R_{2q} + ((2q)^2 - q + 1)T_{2q}, & \text{if } p = 2q \text{ for some integer } q. \end{cases}$$

Note that T, S and T_p, S_p are bases of $H_2(X; \mathbb{Z})$ and $H_2(X_p; \mathbb{Z})$, respectively. The intersection matrices of X , X_{2q-1} and X_{2q} with respect to these bases are

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},$$

respectively. The following algebraic lemma is easily checked, using these bases.

Lemma 3.1. *If $v \in H_2(X_p; \mathbb{Z})$ satisfies either $v \cdot v = -2$ or $v \cdot v = -1$, then $v = \pm S_p$.*

Let g_{-1} (resp. g_p ($p \geq 1$)) be the minimal number of genera of smoothly embedded closed surfaces which represent the class S in $H_2(X; \mathbb{Z})$ (resp. S_p in $H_2(X_p; \mathbb{Z})$).

Lemma 3.2. (1) *For each non-negative integer p , there exists a positive integer N_p such that $g_{2q-1} > g_{2p-1}$ for any integer $q > N_p$.*

(2) *For each positive integer p , there exists a positive integer N'_p such that $g_{2q} > g_{2p}$ for any integer $q > N'_p$.*

Proof. (1) Fix a positive integer p . Let N_p be a positive integer satisfying the condition $2g_{2p-1} - 2 \leq 2N_p - 2$. Now assume $q > N_p$. By the version of the adjunction inequality of [2] we get

$$2g_{2q-1} - 2 \geq |-1 - (2q-1)| - 2.$$

The assumption of q implies $g_{2q-1} > g_{2p-1}$. The case (2) is similar. \square

Corollary 3.3. *Let p be a positive integer. Then the following hold.*

- (1) X_{2q-1} is homeomorphic but not diffeomorphic to X_{2p-1} or X , for any $q > N_p$.
- (2) X_{2q} is homeomorphic but not diffeomorphic to X_{2p} for any $q > N'_p$.

Proof. (1) Assume $q > N_p$, and there exists a diffeomorphism $f : X_{2q-1} \rightarrow X_{2p-1}$. Lemma 3.1 shows $f_*(S_{2q-1}) = \pm S_{2p-1}$. We thus have $g_{2q-1} = g_{2p-1}$, which contradicts Lemma 3.2. The claim thus follows from Lemma 2.1. The same argument holds for the X case. The (2) case is similar. \square

To prove the main theorem, we use the following fact, which was kindly pointed to us by Wendl.

Lemma 3.4 (Wendl [15]). *For any closed connected oriented 3-manifold, it has at most finitely many different strongly fillable contact structures up to isomorphisms.*

Proof. Theorem 0.6 in [10] tells that for any given non-negative integer, every closed connected 3-manifold has at most finitely many contactomorphism classes of tight contact structures with Giroux torsion equal to that integer. The claim thus follows from Corollary 3 in [13] which says that a contact structure with Giroux torsion > 0 is not strongly fillable. \square

Since a Stein filling is a strong filling, this lemma and the above corollary imply the following, which shows the main theorem.

Corollary 3.5. (1) *There exists a contact structure ξ on the boundary ∂X such that at least infinitely many of X_{2p-1} 's ($p \geq 1$) are Stein fillings of $(\partial X, \xi)$ and that these fillings are all homeomorphic but mutually non-diffeomorphic.*
 (2) *There exists a contact structure η on the boundary ∂X such that at least infinitely many of X_{2p} 's ($p \geq 1$) are Stein fillings of $(\partial X, \eta)$ and that these fillings are all homeomorphic but mutually non-diffeomorphic.*

Remark 3.6. (1) Corollary 3.5.(1) and (2) give spin and non-spin examples, respectively.

(2) This construction clearly has many variations. For example, we can change knot types and framings of the attaching circles of 2-handles of X attached to $T^2 \times D^2$. A more general construction in the $b_2 \geq 2$ case and the non-simply connected case will be discussed in [17] applying techniques in [16].

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